

# COMP4434 Big Data Analytics

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## Lecture 7 Dimensionality Reduction

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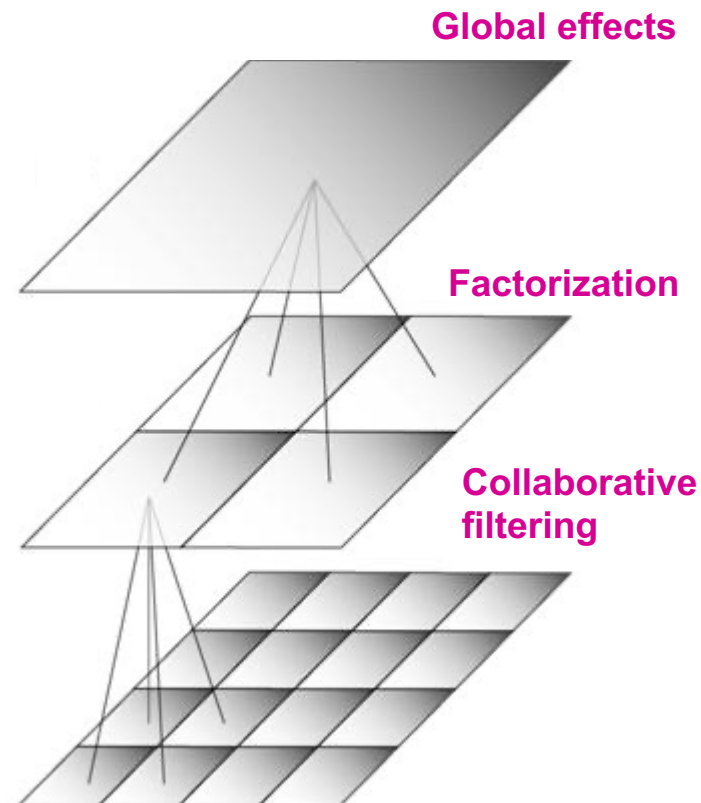
# BellKor Recommender System

- **The winner of the Netflix Challenge!**

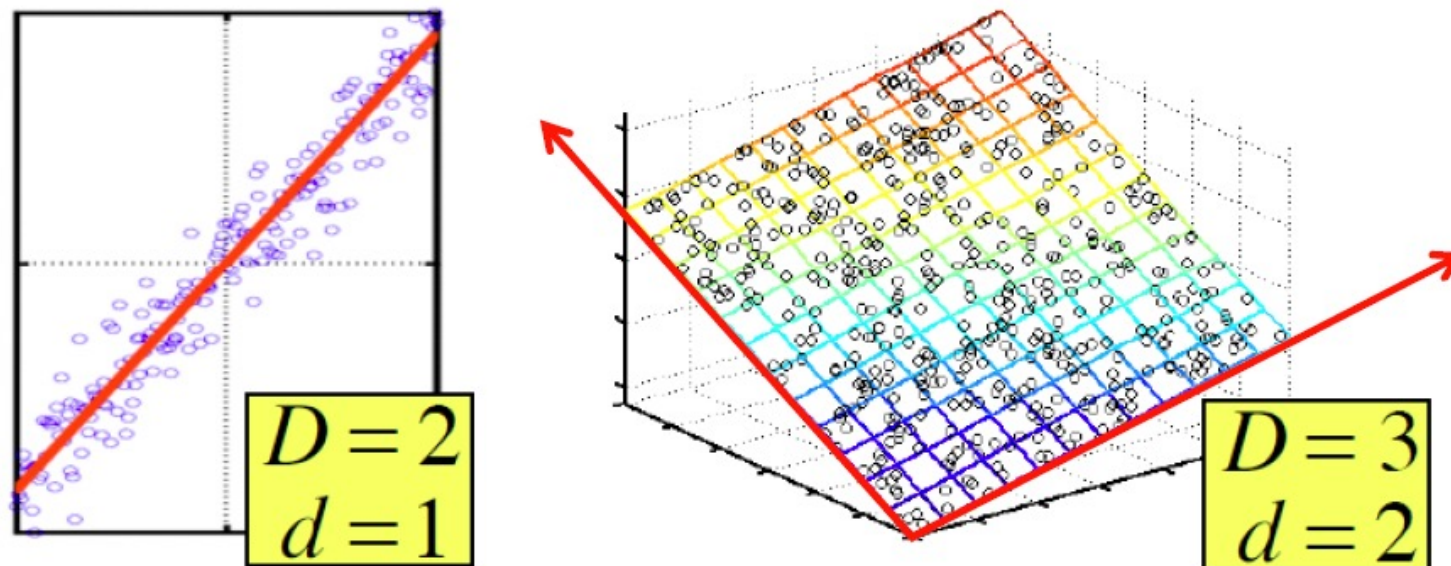
- **Multi-scale modeling of the data:**

Combine top level, “regional” modeling of the data, with a refined, local view:

- **Global:**
  - Overall deviations of users/movies
- **Factorization:**
  - Addressing “regional” effects
- **Collaborative filtering:**
  - Extract local patterns



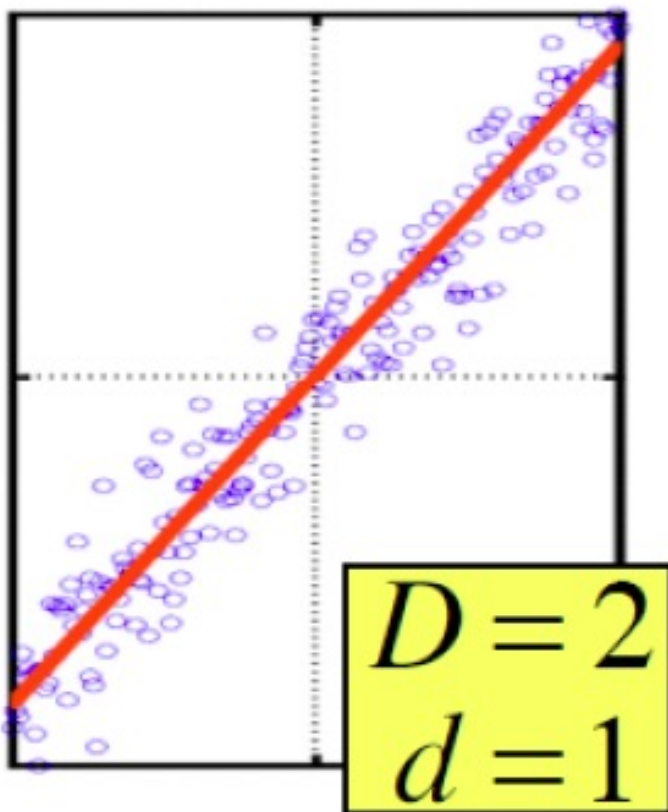
# Dimensionality Reduction



- **Assumption:** Data lies on or near a low  $d$ -dimensional subspace
- **Red axes of this subspace are effective representation of the data**

# Dimensionality Reduction

- Goal of dimensionality reduction is to discover the red axis of data!



Rather than representing every point with 2 coordinates we represent each point with 1 coordinate (corresponding to the position of the point on the red line).

By doing this we incur a bit of **error** as the points do not exactly lie on the line

# Dimensionality Reduction

- Compress / reduce dimensionality:

- E.g.,

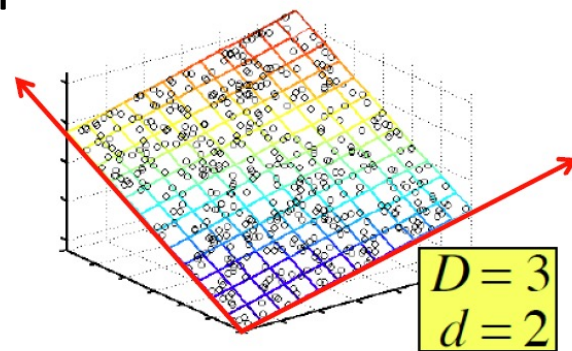
| customer | day | We<br>7/10/96 | Th<br>7/11/96 | Fr<br>7/12/96 | Sa<br>7/13/96 | Su<br>7/14/96 |
|----------|-----|---------------|---------------|---------------|---------------|---------------|
| ABC Inc. |     | 1             | 1             | 1             | 0             | 0             |
| DEF Ltd. |     | 2             | 2             | 2             | 0             | 0             |
| GHI Inc. |     | 1             | 1             | 1             | 0             | 0             |
| KLM Co.  |     | 5             | 5             | 5             | 0             | 0             |
| Smith    |     | 0             | 0             | 0             | 2             | 2             |
| Johnson  |     | 0             | 0             | 0             | 3             | 3             |
| Thompson |     | 0             | 0             | 0             | 1             | 1             |

The above matrix is really “2-dimensional.” All rows can be reconstructed by scaling  $[1\ 1\ 1\ 0\ 0]$  or  $[0\ 0\ 0\ 1\ 1]$

- $10^6$  rows;  $10^3$  columns; no updates
- Random access to any cell(s); **small error: OK**

# Why Reduce Dimensions?

- **Data preprocessing is an important part for effective machine learning and data mining**
  - ML and DM techniques may not be effective for high-dimensional data
- **Dimensionality reduction is an effective approach to downsizing data**
  - The intrinsic dimension may be small
  - Discover hidden correlations/topics  
E.g., words that occur commonly together
  - Remove redundant and noisy features  
E.g., not all words are useful
- **Interpretation and visualization**
- **Easier storage and processing of the data**



# Rank of a Matrix

- What is **rank** of a matrix **A**?
  - Number of **linearly independent** columns of **A**
  - E.g., Matrix **A** =  $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$  has rank **r=2**
    - **Why?** The first two rows are linearly independent, so the rank is at least 2, but all three rows are linearly dependent (the first is equal to the sum of the second and third) so the rank must be less than 3.
- **Why do we care about low rank?**
  - We can write **A** as two “basis” vectors:  $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \end{bmatrix}$
  - And new coordinates of :  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

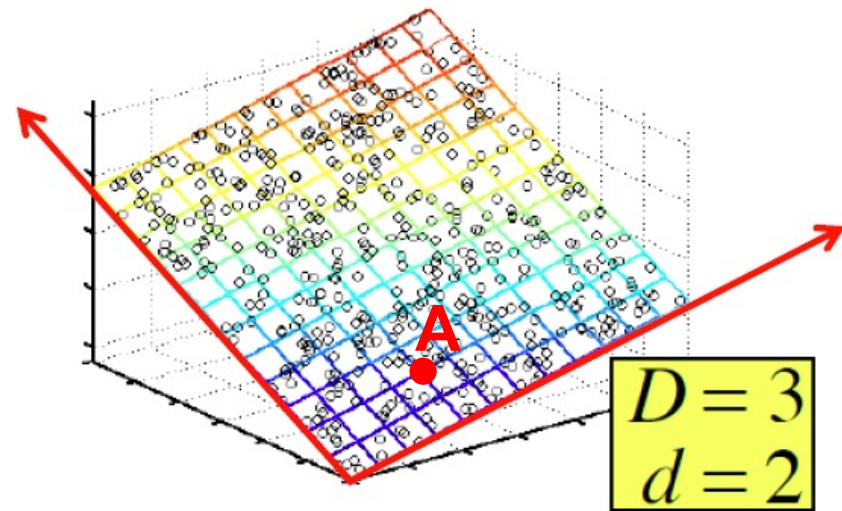
# Rank is “Dimensionality”

- **Cloud of points 3D space:**

- Think of point positions as a matrix:

1 row per point:

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$



- **We can rewrite coordinates more efficiently!**

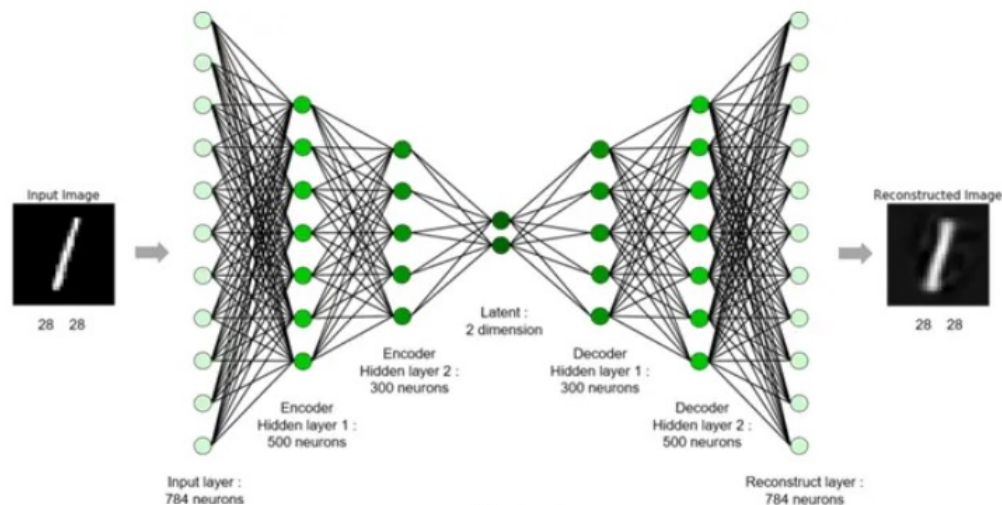
- Old basis vectors:  $[1 \ 0 \ 0]$   $[0 \ 1 \ 0]$   $[0 \ 0 \ 1]$
- **New basis vectors:**  $[1 \ 2 \ 1]$   $[-2 \ -3 \ 1]$
- Then **A** has new coordinates:  $[1 \ 0]$ . **B**:  $[0 \ 1]$ , **C**:  $[1 \ -1]$ 
  - Notice: We reduced the number of coordinates!



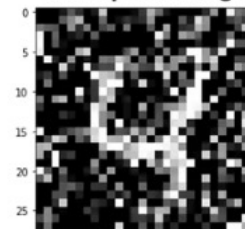
# Dimensionality Reduction Techniques

- Singular value decomposition (SVD)
- Principal component analysis (PCA)
- Non-negative matrix factorization (NMF)
- Linear discriminant analysis (LDA)
- Autoencoders
- Feature selection

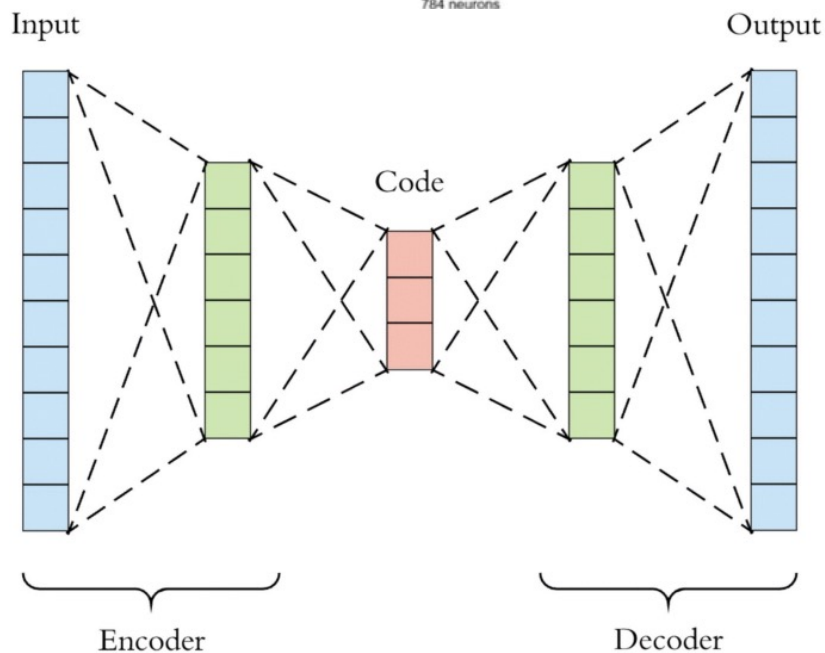
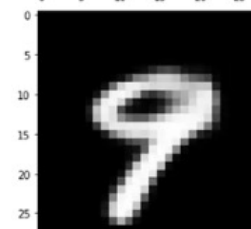
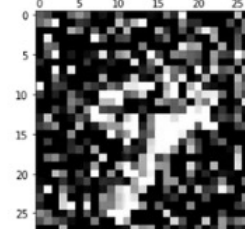
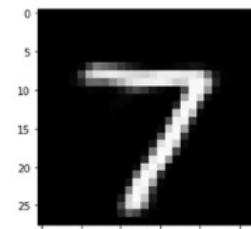
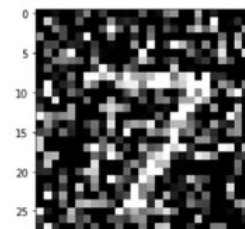
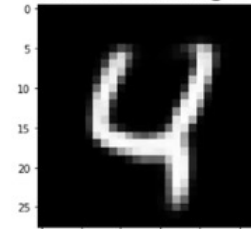
# Dimensionality Reduction by using Hidden Layers



Corrupted image



Predicted image



# In CF, feature learning is Dimensionality Reduction

| Movie                     | Alice<br>$\theta^{(1)}$ | Bob<br>$\theta^{(2)}$ | Carol<br>$\theta^{(3)}$ | Dave<br>$\theta^{(4)}$ | X1 | X2 |
|---------------------------|-------------------------|-----------------------|-------------------------|------------------------|----|----|
| Love letter $x^{(1)}$     | 5                       | 5                     | 0                       | 0                      | ?  | ?  |
| Romancer $x^{(2)}$        | 5                       | ?                     | ?                       | 0                      | ?  | ?  |
| Stay with me $x^{(3)}$    | ?                       | 4                     | 0                       | ?                      | ?  | ?  |
| KungFu Panda $x^{(4)}$    | 0                       | 0                     | 5                       | 4                      | ?  | ?  |
| FightFightFight $x^{(5)}$ | 0                       | 0                     | 5                       | ?                      | ?  | ?  |

- Hypothesis function

$$h_{i,j}(x, \theta) = (\theta^{(j)})^T x^{(i)} = \theta_0^{(j)} x_0^{(i)} + \theta_1^{(j)} x_1^{(i)} + \theta_2^{(j)} x_2^{(i)} + \theta_3^{(j)} x_3^{(i)}$$

- In CF, the model learns feature X1 & X2.
- Represent each movie as its ratings & reduce the dimension from 4 (# users) to 2 (X1 & X2)

# Can we learn latent factors directly?

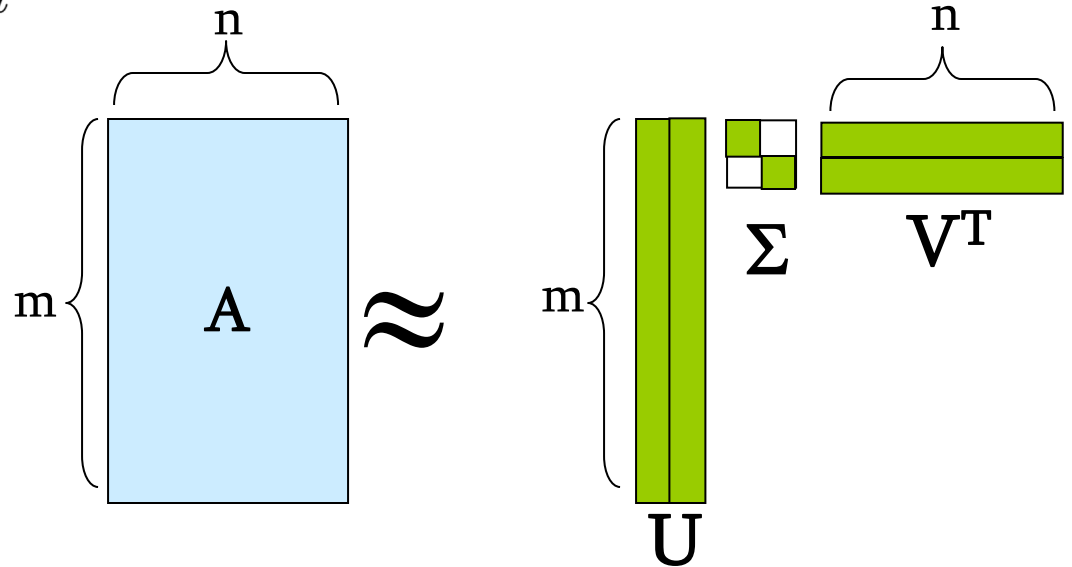
$$\text{SVD: } A = U \Sigma V^T$$

$$\text{Netflix data: } R \approx Q \cdot P^T$$

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# Singular value decomposition (SVD) - Properties

$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i$$



It is **always** possible to decompose a real matrix  $\mathbf{A}$  into  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ , where

- $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ : **unique**
- $\mathbf{U}, \mathbf{V}$ : **column orthonormal**
  - $\mathbf{U}^T \mathbf{U} = \mathbf{I}; \mathbf{V}^T \mathbf{V} = \mathbf{I}$  ( $\mathbf{I}$ : identity matrix)
  - (Columns are orthogonal unit vectors)
- $\mathbf{\Sigma}$ : **diagonal**
  - Entries (**singular values**) are **positive**, and sorted in decreasing order ( $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ )

# Singular value decomposition (SVD) - Definition

$$\mathbf{A}_{[m \times n]} = \mathbf{U}_{[m \times r]} \Sigma_{[r \times r]} (\mathbf{V}_{[n \times r]})^T$$

- **A: Input data matrix**
  - $m \times n$  matrix (e.g.,  $m$  documents,  $n$  terms)
- **U: Left singular vectors**
  - $m \times r$  matrix ( $m$  documents,  $r$  concepts)
- **$\Sigma$ : Singular values**
  - $r \times r$  diagonal matrix (strength of each 'concept')  
( $r$  : rank of the matrix **A**)
- **V: Right singular vectors**
  - $n \times r$  matrix ( $n$  terms,  $r$  concepts)

# SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$

- $U$ : latent representation of movies
- $V$ : latent representation of users

$$\begin{array}{c} \uparrow \\ \text{SciFi} \\ \downarrow \\ \uparrow \\ \text{Romance} \\ \downarrow \end{array}
 \begin{bmatrix} \text{Alice} & \text{Bob} & \text{Carol} & \text{Dave} & \vdots \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}
 =
 \begin{array}{c} \text{SciFi-concept} \\ \text{Romance-concept} \end{array}
 \begin{bmatrix} 0.14 & 0.02 & 0.01 \\ 0.41 & 0.07 & 0.03 \\ 0.55 & 0.09 & 0.04 \\ 0.69 & 0.11 & 0.05 \\ 0.15 & -0.59 & -0.65 \\ 0.07 & -0.73 & 0.68 \\ 0.07 & -0.30 & -0.33 \end{bmatrix}
 \times
 \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix}
 \times
 \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.13 & -0.03 & 0.13 & -0.70 & -0.70 \\ 0.41 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

Columns are orthogonal unit vectors:

$$0.14^2 + 0.41^2 + 0.55^2 + 0.69^2 + 0.15^2 + 0.07^2 + 0.07^2 \approx 1$$

# Only Keep Major Factors

Set smallest singular values to zero

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} 0.14 & 0.02 & -0.01 \\ 0.41 & 0.07 & -0.03 \\ 0.55 & 0.09 & -0.04 \\ 0.69 & 0.11 & -0.05 \\ 0.15 & -0.59 & 0.65 \\ 0.07 & -0.73 & -0.68 \\ 0.07 & -0.29 & 0.32 \end{bmatrix} \times \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.13 & -0.02 & 0.12 & -0.70 & -0.70 \\ 0.41 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$



# Dimensionality Reduction can reduce noise

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} 0.14 & 0.02 \\ 0.41 & 0.07 \\ 0.55 & 0.09 \\ 0.69 & 0.11 \\ 0.15 & -0.59 \\ 0.07 & -0.73 \\ 0.07 & -0.29 \end{bmatrix} \times \begin{bmatrix} 12.4 & 0 \\ 0 & 9.5 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.13 & -0.02 & 0.12 & -0.70 & -0.70 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.94 & 1.01 & 0.94 & -0.00 & -0.00 \\ 2.98 & 3.04 & 2.98 & -0.00 & -0.00 \\ 3.98 & 4.05 & 3.98 & -0.00 & -0.00 \\ 4.97 & 5.06 & 4.97 & -0.01 & -0.01 \\ 0.36 & 1.29 & 0.36 & 4.08 & 4.08 \\ -0.37 & 0.73 & -0.37 & 4.92 & 4.92 \\ 0.18 & 0.65 & 0.18 & 2.04 & 2.04 \end{bmatrix}$$

# Estimate unknown ratings as inner-products of factors

$$? = 2.4$$

users

items

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 |   | 3 |   |   | 5 |   |   | 5 |   | 4 |   |
|   |   | 5 |   | ? | 4 |   |   | 2 | 1 | 3 |   |
| 2 | 4 |   | 1 | 2 |   | 3 |   | 4 | 3 | 5 |   |
|   | 2 | 4 |   | 5 |   |   | 4 |   |   | 2 |   |
|   |   | 4 | 3 | 4 | 2 |   |   |   |   | 2 | 5 |
| 1 |   | 3 |   | 3 |   |   | 2 |   |   | 4 |   |

~

~

items

|     |     |    |
|-----|-----|----|
| .1  | -.4 | .2 |
| -.5 | .6  | .5 |
| -.2 | .3  | .5 |
| 1.1 | 2.1 | .3 |
| -.7 | 2.1 | -2 |
| -1  | .7  | .3 |

•

users

|     |     |    |     |     |     |     |     |     |     |     |     |
|-----|-----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1.1 | -.2 | .3 | .5  | -2  | -.5 | .8  | -.4 | .3  | 1.4 | 2.4 | -.9 |
| -.8 | .7  | .5 | 1.4 | .3  | -1  | 1.4 | 2.9 | -.7 | 1.2 | -.1 | 1.3 |
| 2.1 | -.4 | .6 | 1.7 | 2.4 | .9  | -.3 | .4  | .8  | .7  | -.6 | .1  |

A rank-3 SVD approximation

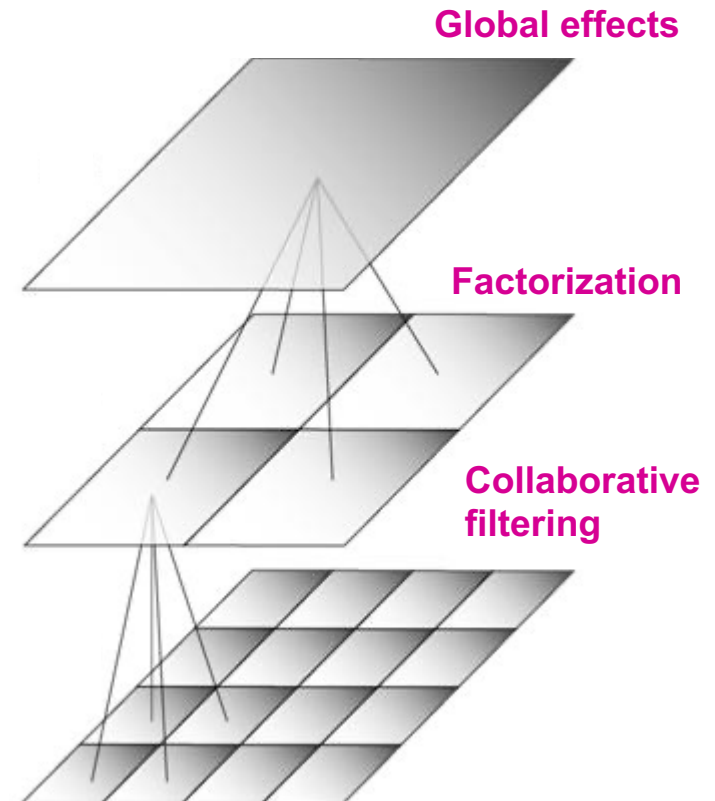
# Recap: BellKor Recommender System

- **The winner of the Netflix Challenge!**

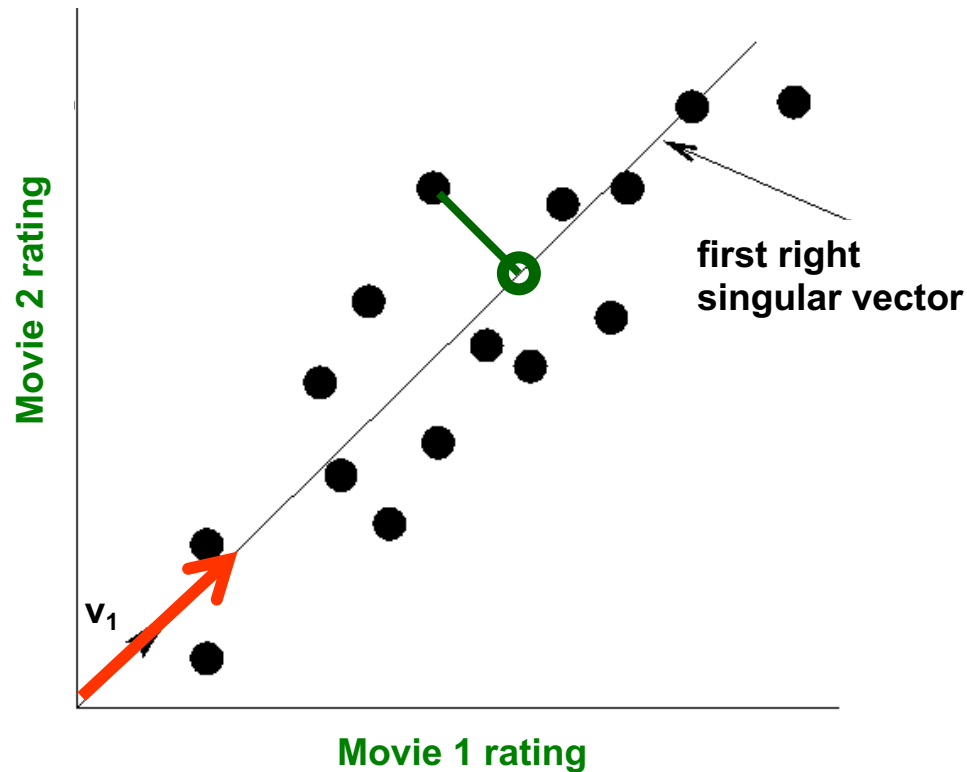
- **Multi-scale modeling of the data:**

Combine top level, “regional” modeling of the data, with a refined, local view:

- **Global:**
  - Overall deviations of users/movies
- **Factorization:**
  - Addressing “regional” effects
- **Collaborative filtering:**
  - Extract local patterns



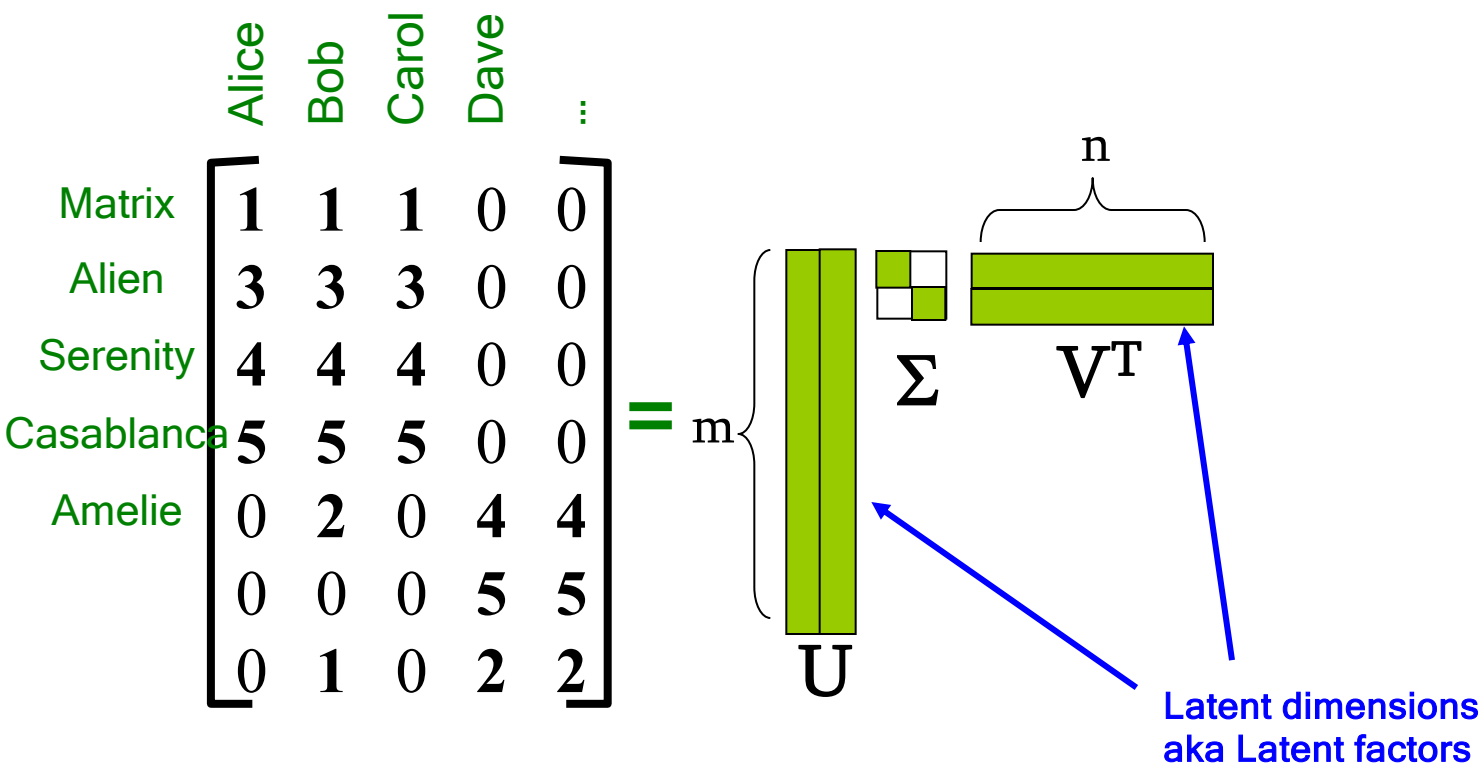
# From 2D to 1D



- Instead of using two coordinates ( $x, y$ ) to describe point locations, let's use only one coordinate ( $z$ )
- Point's position is its location along vector  $v_1$

# Movie-to-User Example: from 5D to 3D

■  $A = U \Sigma V^T$



# Meaning of Singular values

- The first feature is the most important one
- Singular values represent the importance of features

Diagram illustrating the meaning of singular values in the context of feature importance.

The matrix on the left represents the input data, with rows labeled by movie genres (SciFi, Romnce) and columns by users (Alice, Bob, Carol, Dave, ...).

The matrix in the middle represents the singular value decomposition (SVD) components, with the first column labeled "SciFi-concept".

The matrix on the right represents the singular values, with the first value (12.4) circled and labeled "strength" of the SciFi-concept.

The equation shows the input matrix equals the product of the SVD components and the singular values matrix.

$$\begin{bmatrix}
 \text{SciFi} \\
 \text{Romnce}
 \end{bmatrix}
 \begin{bmatrix}
 \text{Alice} & \text{Bob} & \text{Carol} & \text{Dave} & \vdots \\
 1 & 1 & 1 & 0 & 0 \\
 3 & 3 & 3 & 0 & 0 \\
 4 & 4 & 4 & 0 & 0 \\
 5 & 5 & 5 & 0 & 0 \\
 0 & 2 & 0 & 4 & 4 \\
 0 & 0 & 0 & 5 & 5 \\
 0 & 1 & 0 & 2 & 2
 \end{bmatrix}
 =
 \begin{bmatrix}
 \text{SciFi-concept} \\
 0.14 & 0.02 & -0.01 \\
 0.41 & 0.07 & -0.03 \\
 0.55 & 0.09 & -0.04 \\
 0.69 & 0.11 & -0.05 \\
 0.15 & -0.59 & 0.65 \\
 0.07 & -0.73 & -0.68 \\
 0.07 & -0.29 & 0.33
 \end{bmatrix}
 \times
 \begin{bmatrix}
 12.4 & 0 & 0 \\
 0 & 9.5 & 0 \\
 0 & 0 & 1.3
 \end{bmatrix}
 \times
 \begin{bmatrix}
 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\
 0.13 & -0.02 & 0.13 & -0.70 & -0.70 \\
 0.41 & -0.80 & 0.40 & 0.09 & 0.09
 \end{bmatrix}$$

# Singular values also represent the variance

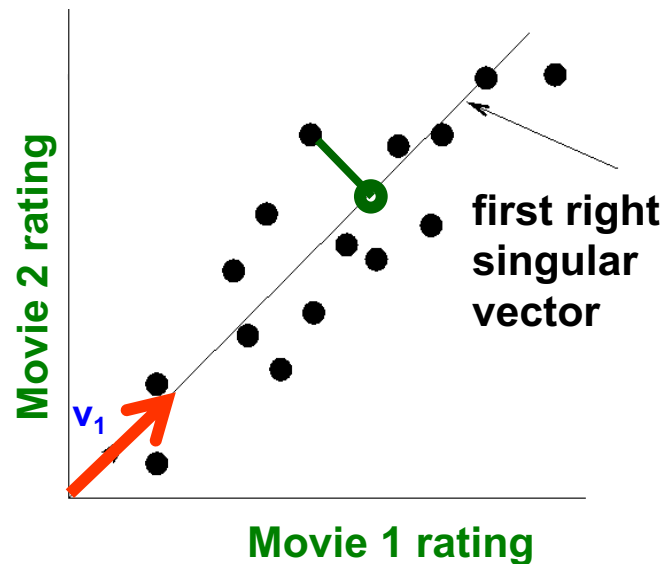
## ■ $A = U \Sigma V^T$ - example:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.14 & 0.02 & -0.01 \\ 0.41 & 0.07 & -0.03 \\ 0.55 & 0.09 & -0.04 \\ 0.69 & 0.11 & -0.05 \\ 0.15 & -0.59 & 0.65 \\ 0.07 & -0.73 & -0.68 \\ 0.07 & -0.29 & 0.32 \end{bmatrix} \times$$

$$\begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.13 & -0.02 & 0.12 & -0.70 & -0.70 \\ 0.41 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

variance ('spread')  
on the  $v_1$  axis



# SVD: minimizing reconstruction errors

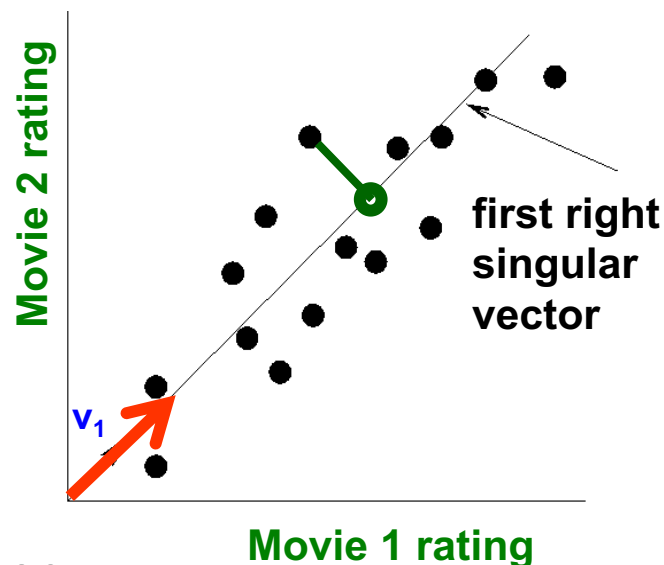
- How to choose  $v_1$ ?  
Minimize reconstruction error
- Goal: Minimize the sum of reconstruction errors:

$$\sum_{i=1}^N \sum_{j=1}^D \|a_{ij} - b_{ij}\|^2$$

- where  $a_{ij}$  are the original values in matrix  $\mathbf{A}$  and  $b_{ij}$  are the reconstructed ones

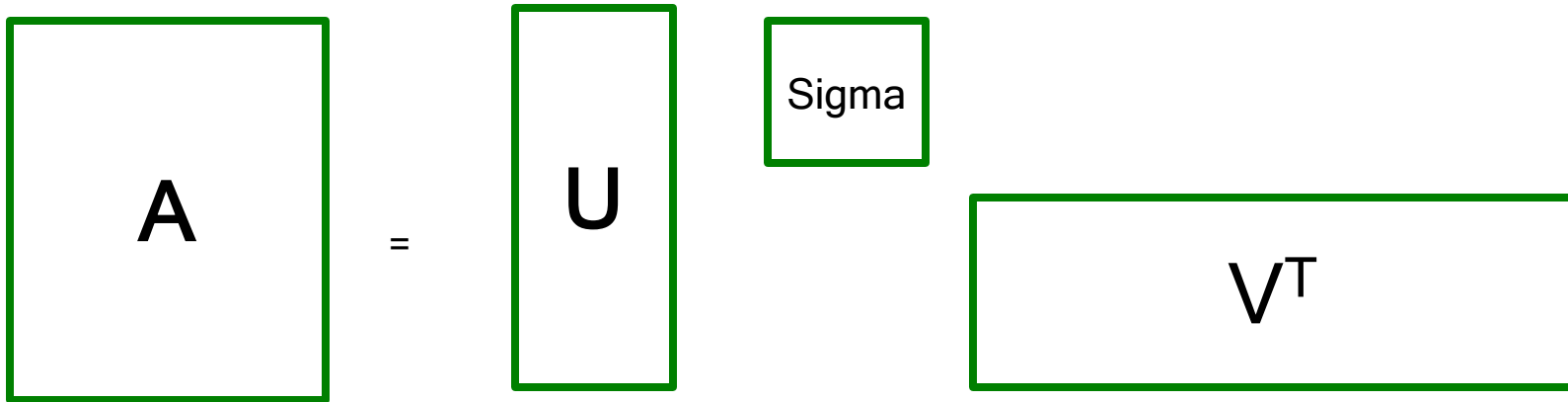
**SVD gives ‘best’ axis to project on:**

- ‘best’ = minimizing the reconstruction errors
- In other words, **minimum reconstruction error**

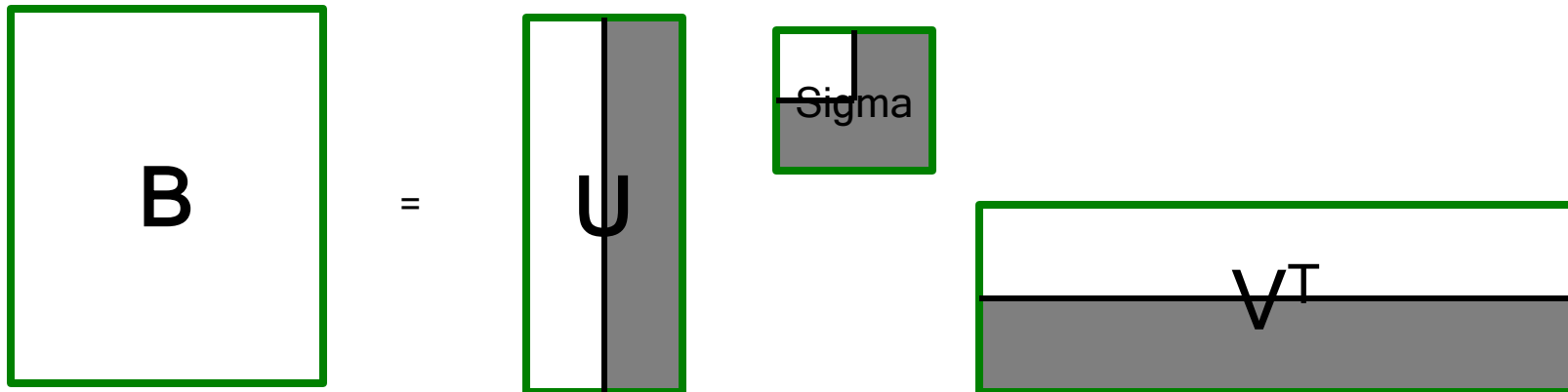




# SVD – Best Low Rank Approx.



**B is best approximation of A**



# SVD – Best Low Rank Approx.

- Theorem:

Let  $A = U \Sigma V^T$  and  $B = U S V^T$  where

$S = \text{diagonal } r \times r \text{ matrix}$  with  $s_i = \sigma_i$  ( $i=1 \dots k$ ) else  $s_i=0$

then  $B$  is a **best**  $\text{rank}(B)=k$  approx. to  $A$

What do we mean by “best”:

- $B$  is a solution to  $\min_B \|A-B\|_F$  where  $\text{rank}(B)=k$

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}_{m \times n} = \begin{pmatrix} u_{11} & \dots & & \\ \vdots & \ddots & & \\ u_{m1} & & & \end{pmatrix}_{m \times r} \begin{pmatrix} \sigma_{11} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & \end{pmatrix}_{r \times r} \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \\ & & \end{pmatrix}_{r \times n}$$

$$\|A - B\|_F = \sqrt{\sum_{ij} (A_{ij} - B_{ij})^2}$$

# Users-to-Movies Example

- Best approximation for rank = 2
- $\|A-B\|_F = \sqrt{\sum_{ij} (A_{ij}-B_{ij})^2}$  is minimum

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} 0.94 & 1.01 & 0.94 & -0.00 & -0.00 \\ 2.98 & 3.04 & 2.98 & -0.00 & -0.00 \\ 3.98 & 4.05 & 3.98 & -0.00 & -0.00 \\ 4.97 & 5.06 & 4.97 & -0.01 & -0.01 \\ 0.36 & 1.29 & 0.36 & 4.08 & 4.08 \\ -0.37 & 0.73 & -0.37 & 4.92 & 4.92 \\ 0.18 & 0.65 & 0.18 & 2.04 & 2.04 \end{bmatrix}$$

Frobenius norm:

$$\|M\|_F = \sqrt{\sum_{ij} M_{ij}^2}$$

# SVD - Complexity

- $\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^T$ : **unique**
- SVD: picks up linear correlations
- **To compute SVD:**
  - $O(nm^2)$  or  $O(n^2m)$  (whichever is less)
- **But:**
  - Less work, if we just want singular values
  - or if we want first  $k$  singular vectors
  - or if the matrix is sparse
- **Implemented in** linear algebra packages like
  - LINPACK, Matlab, SPlus, Mathematica ...
  - `numpy.linalg.svd` in Python