A Polynomial Kernel for Diamond-Free Editing

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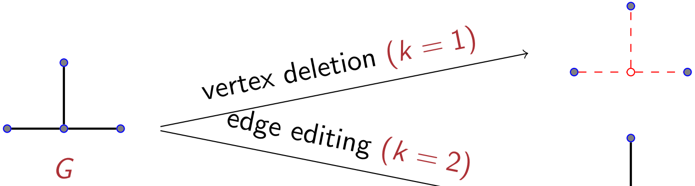
Workshop on Kernelization　　D-Day, 2019

Joint work with Ashutosh Rai, R. B. Sandeep, Junjie Ye

Graph modification problems

Graph modification problem for property P

Input: A graph G and an integer k.
Task: Can we apply $\leq k$ modifications to G to make it satisfy P?



P: being a cluster graph.

Three questions

1. Is it NP-complete?
2. Can it be solved in time $f(k) \cdot n^{O(1)}$; if yes, what is the (asymptotically) best f?
3. Does it have a polynomial kernel?

Theorem (Lewis and Yannakakis 80).
For a hereditary property, the vertex deletion problem is either NP-complete or trivial.

Such a general result for edge modification problems is unknown, and very unlikely.

(Complexity Status of Edge Modification Problems@FPT WIKI)

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Kernelization

An algorithm maps (I, k) into (I', k') in time polynomial in $|I| + k$ such that

- (I, k) is a yes-instance iff (I', k') is a yes-instance, and
- $|I'| + k' \leq g(k)$ for some computable function g.

The function g is the size of the kernel, and it is a polynomial kernel if g is polynomial.

H-free modification problems

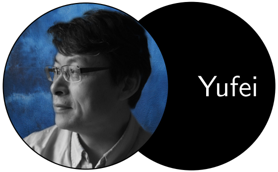
$|V(H)| \geq 2$

The "simplest" property is H-free—not containing H as an induced subgraph.
(For example, H is P_3 for cluster graphs.)

	vertex deletion	edge modification
NPC	Yes [Lewis & Yannakakis 81]	Dichotomy [Sandeep 16]
FPT	Yes [Cai 96]	Yes [Cai 96]
Poly kernel	Yes [Abu-Khzam 10]	?

no for someone

no for a lot



most no...

which has?

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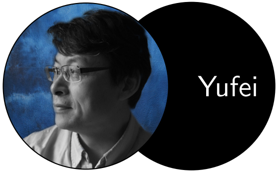


most no...

which has?

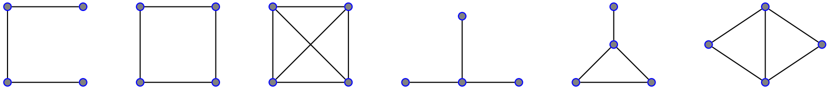
Drange & Mi. Pilipczuk:
"the existence of a polynomial kernel for any of H-Free Editing, H-Free Edge Deletion, or H-Free Completion problems is in fact a very rare phenomenon, and basically happens only for specific, constant-size graphs H."

What are known

H	completion	deletion	editing
$P_{\leq 4}$	yes	yes	yes
$P_{>4}$	no	no	no
$C_{\geq 4}$	no	no	no
K_ℓ	trivial	yes	yes
$K_{\geq 5} - e$	trivial	no	no
3-connected (≥ 2 non-edges)	no	no	no

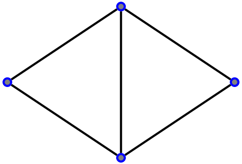
all yes when $|V(H)| = 2$ or 3.

Four-vertex graphs



(a) P_4 (b) C_4 (c) K_4 (d) claw (e) paw (f) diamond

H	completion	deletion	editing
P_4	$O(k^3)$ [Guillemot et al. 13]		
K_4	trivial	$O(k^4)$ [Cai 12]	
diamond	trivial	$O(k^3)$ [Sandeep & Sivadasan 15]	$O(k^8)$ [C et al. 2018]
claw	unknown		
paw	unknown		
C_4	no [Guillemot et al. 13]		



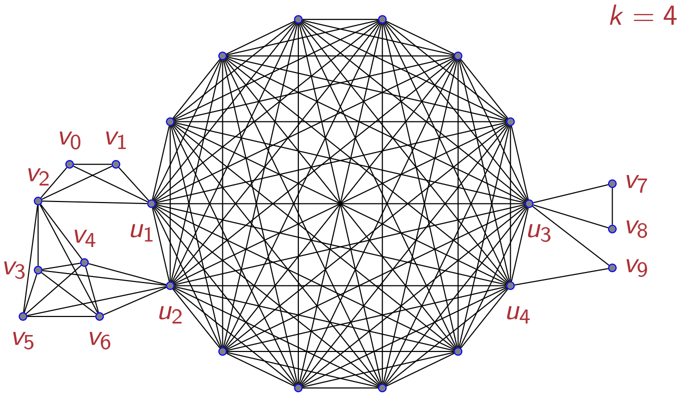
Diamond-free edge editing

Input: A graph G and an integer k.
Task: Can we edit (add/delete) $\leq k$ edges to make G diamond-free?

Main result: an $O(k^8)$-vertex kernel.

Maximal Cliques

An example



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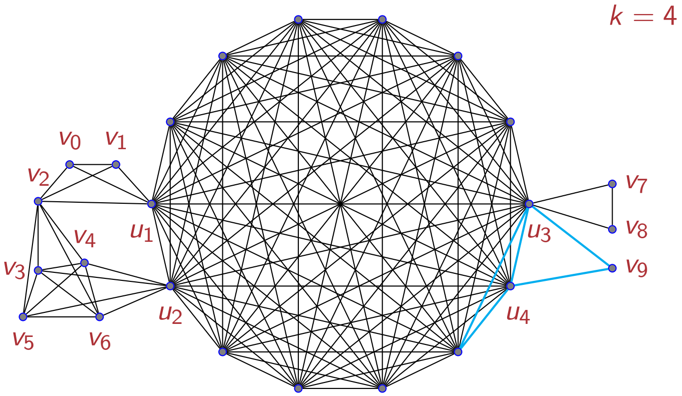
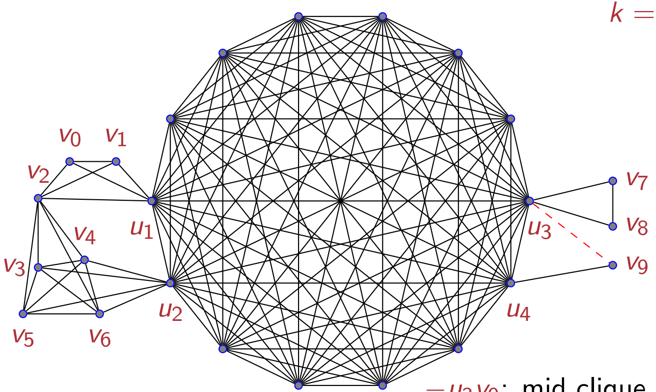
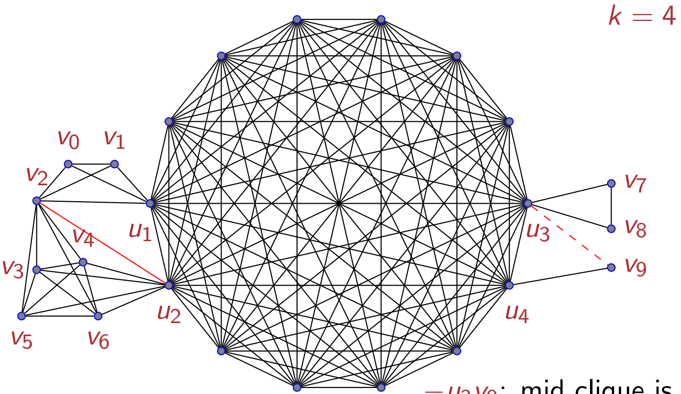




$k = 4$

$-u_3 v_9$: mid clique is too big to fall.



$-u_3 v_9$: mid clique is too big to fall.

$+u_2 v_2$: shared by too many.

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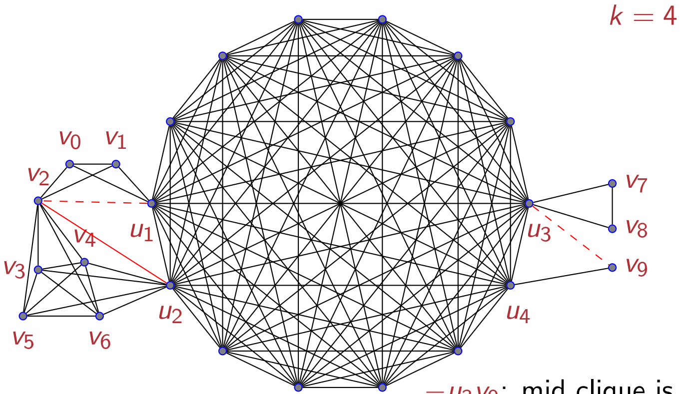

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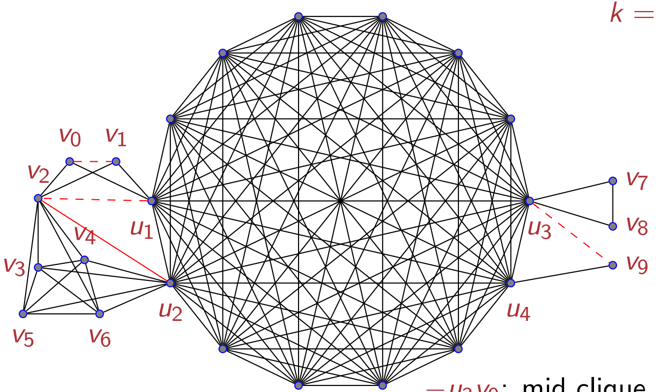

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$-v_0 v_1$: not in "original" diamonds.

Maximal cliques

We are concerned with maximal cliques.

A maximal clique is of

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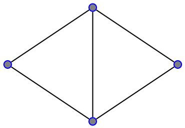

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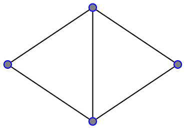
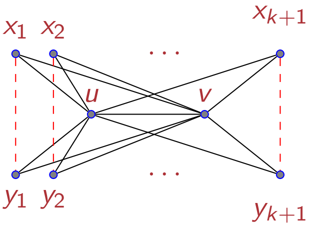


A maximal clique K is *big* if $|K| \geq 3k + 2$, and *small* otherwise.

Basic rules

Rule 1(2).
If an edge (resp. non-edge) uv is the only edge (resp. non-edge) shared by $k+1$ diamonds, then delete (resp. add) uv, and decrease k by one.



We may assume throughout that Rules 1–2 have been exhaustively applied.

Properties: added edges

$E_{\pm}$ ($E_{+} \cup E_{-}$): a minimum solution to G; $G^{*} = G \triangle E_{\pm}$.

Proposition.
Big max cliques of G remain max cliques in G^{*}.

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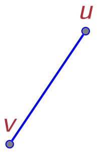

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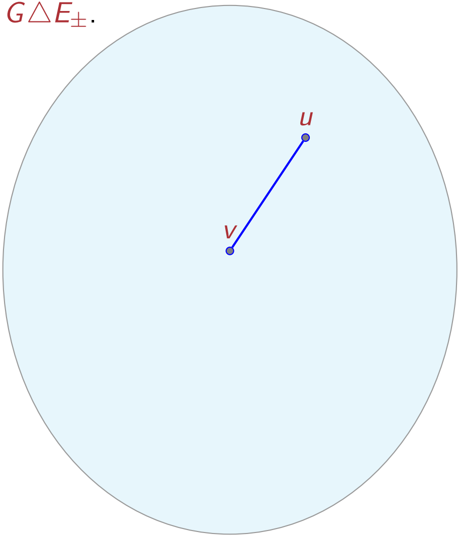

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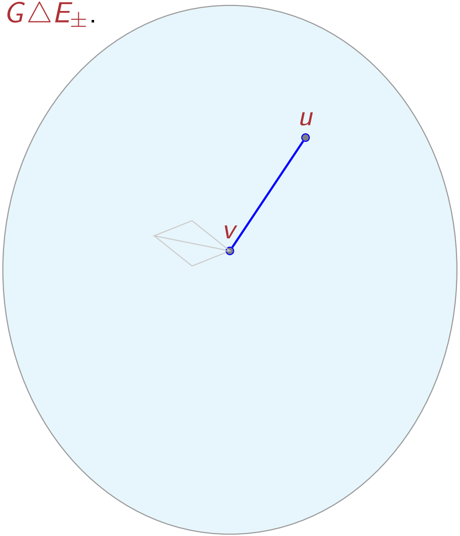
Claim: v is in a diamond in $G[U]$.



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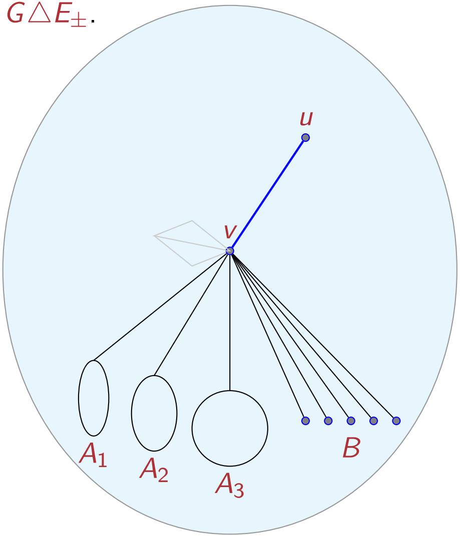

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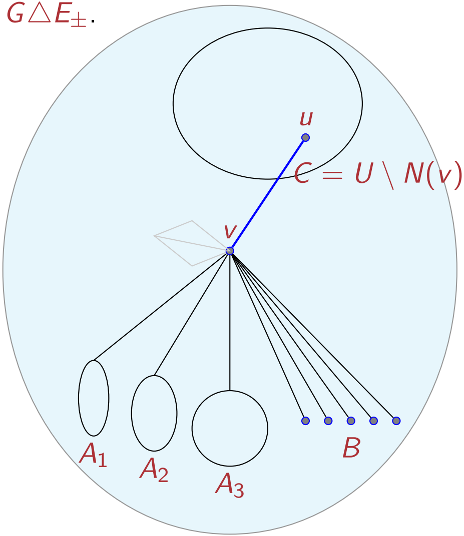
We can find a smaller solution. □



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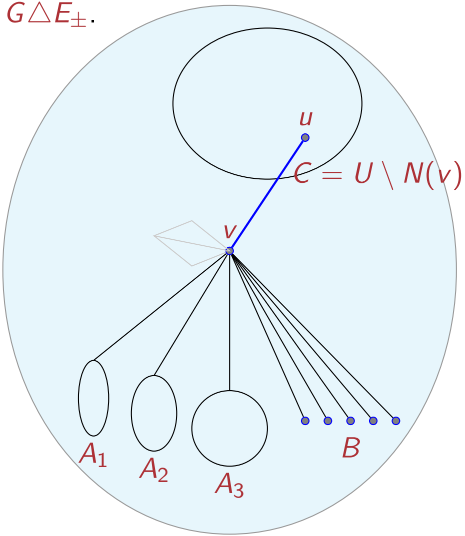

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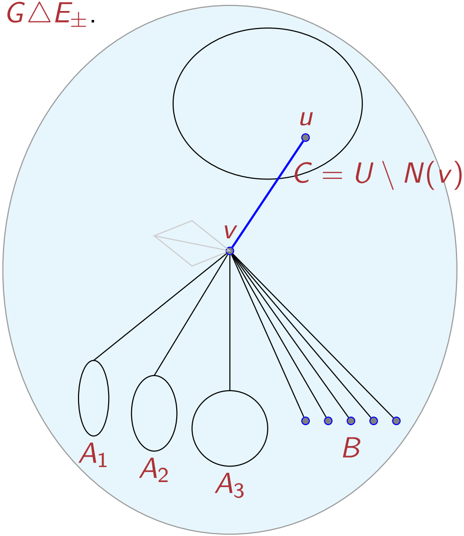

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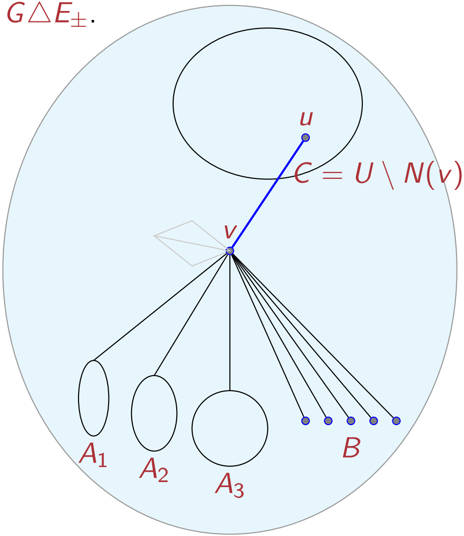
$|K \cap U| > 2$, hence not a max clique of G^{*}.
By the proposition above, K is not big. □



Properties: deleted edges

Lemma.
Let $uv \in E_-$. If a max clique K contains u, v, then K is small.
Moreover, if K is of type II, then K intersects some small type-I clique.

Vulnerable vertices

A vertex v is *vulnerable* if v is in

- some small type-I clique; or
- a type-II clique that intersects some small type-I clique.

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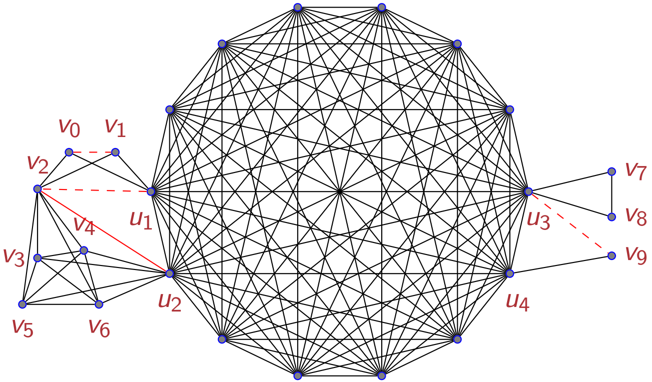

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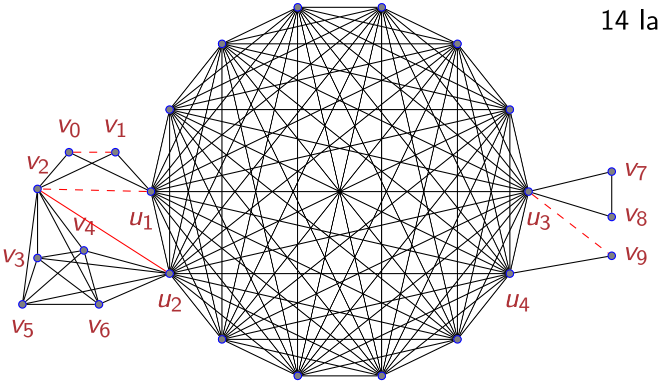

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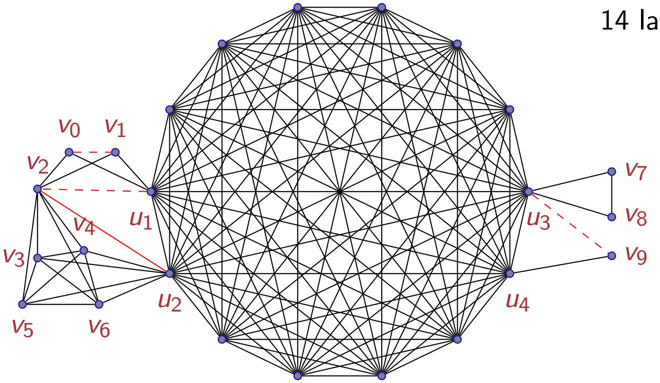

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Lemma. A minimum solution touches only vulnerable vertices.

Main rules

Lemma.
In a yes-instance, at most $18k^3 + 2k$ vertices are in small type-I cliques.

Big type-I cliques

Lemma.
A yes-instance has at most $6k^2$ big type-I cliques.

Rule 3.
If a big type-I clique contains a "private" protected vertex, delete it.
Lemma.
In a yes-instance, each big type-I clique has $O(k^3)$ vertices if Rule 3 is not applicable.

vertices in all type-I cliques: $O(k^3 + k^5) = O(k^5)$.

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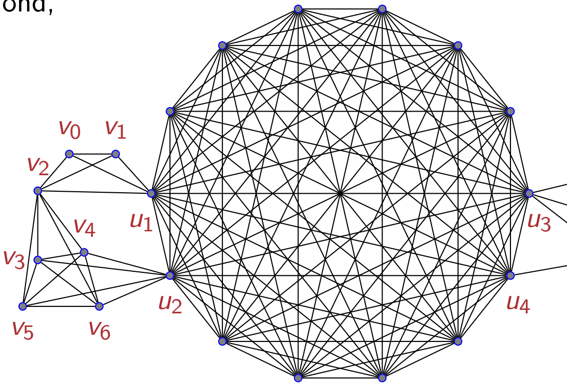
Type-II cliques

Let $T(G)$ denote the vertices that occur *only* in cliques of type II.

Rule 4.
If there is a protected vertex x contained in *only* in type-II cliques, delete it.

The last reduction rule

An edge in a type-II clique K is not in a diamond,



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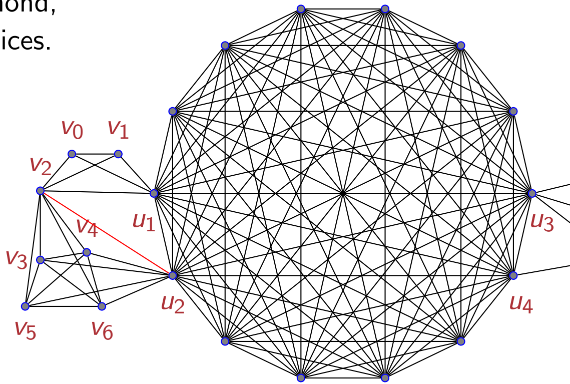

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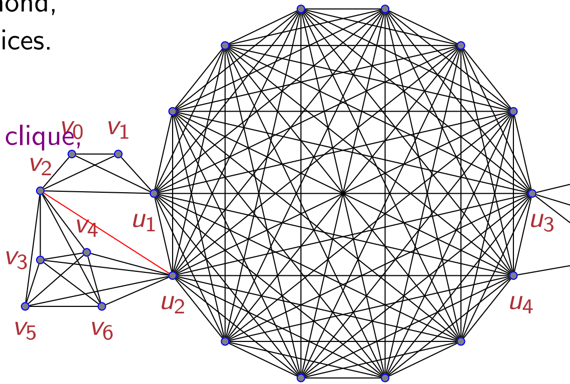

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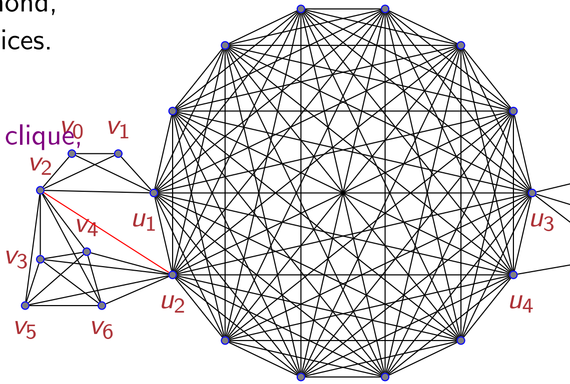

Rule 5. (Informal)

For each pair u, v in small type-I cliques

1. mark $k+1$ common neighbors of u, v;
2. for each marked neighbor w of u, mark $k+1$ common neighbors of u, w.

Delete all unmarked vertices x in $T(G)$.

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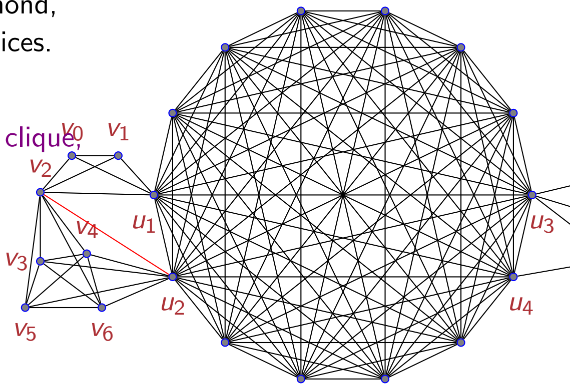

In total, $O(k^8)$ vertices marked.

Putting together

$$O(k^3) + O(k^5) + O(k^8) = O(k^8).$$

The implementation

We do *not* need to enumerate the maximal cliques (exponential number of them).

It is sufficient to find

- all vertices in a small type-I clique,
- all vertices in a type-I clique, and
- all vulnerable vertices.

Key observation: we can enumerate cross edges of all diamonds, from which we can identify all vertices and edges in type-I cliques.

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Diamond-free edge deletion

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Input: A graph G and an integer k.
Task: Can we delete $\leq k$ edges to make G diamond-free?

Rule 2. If there is a vertex not in any small maximal clique, delete it.
Rule 3. Delete all edges and vertices not in any type-I clique.

Lemma.
The diamond-free edge deletion problem has a kernel of $O(k^3)$ vertices.

Michał: claw is difficult.

Can we take paw first (apparently attackable)?

Thanks!

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