

Characterization of Circular-Arc Graphs

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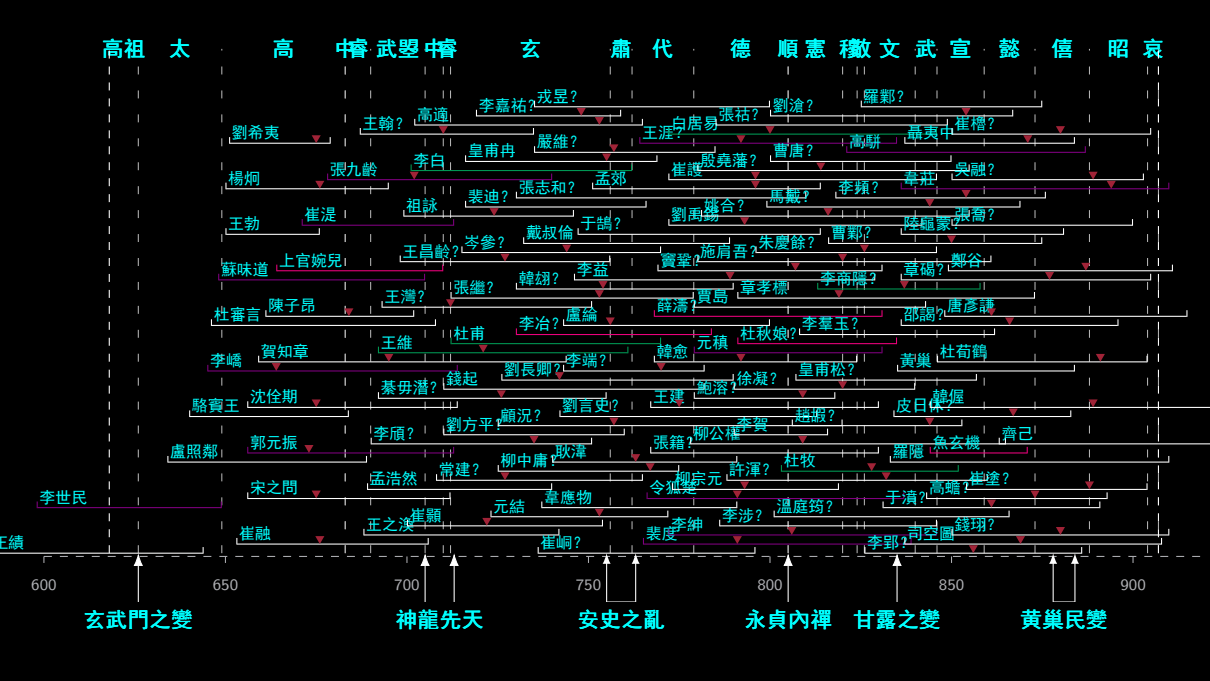
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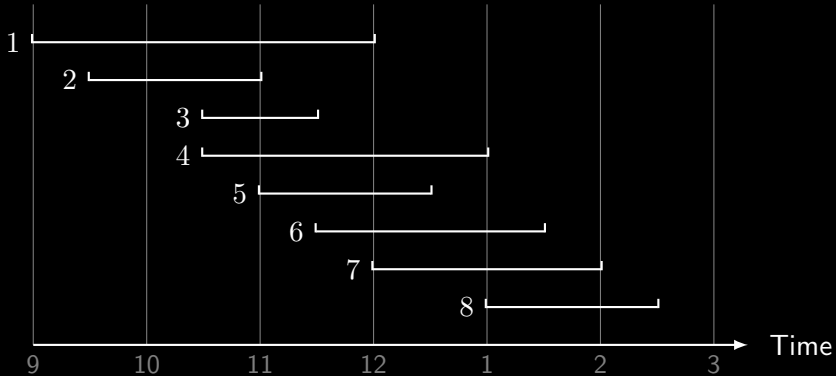
Department of Computing
電子計算學系

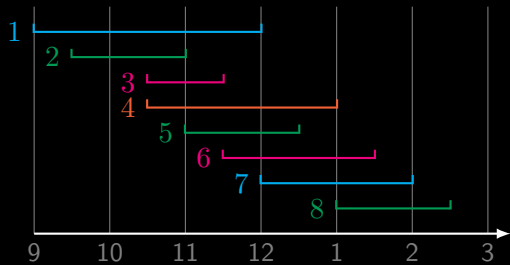
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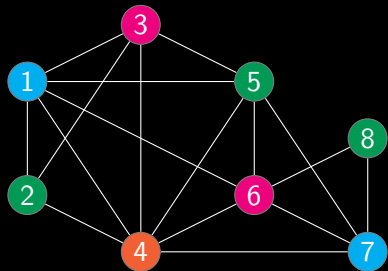


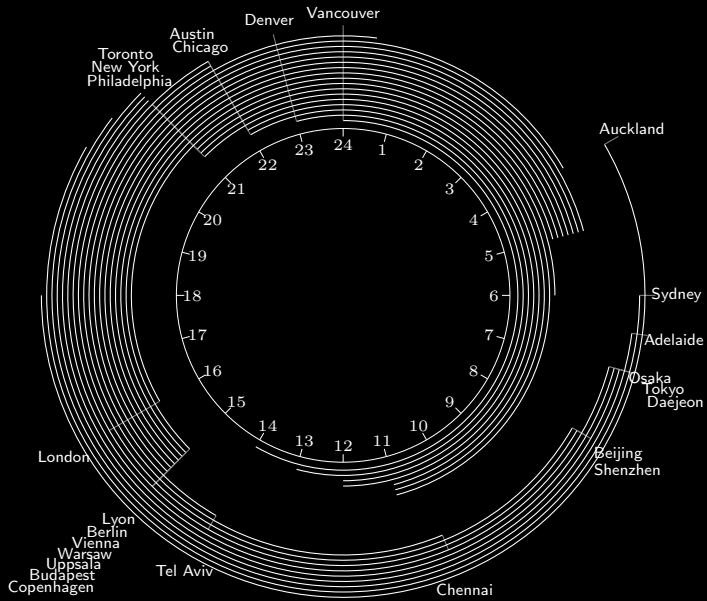
How many meeting rooms do we need to accommodate the following meeting plan?





Time





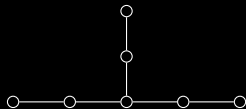
Theorem (Feder, Hell, and Huang 1998, 1999).

The list homomorphism problem (whether an input graph G admits a list homomorphism to a fixed graph H) can be solved in polynomial time if and only if H is an **interval graph** (for reflexive graphs) or a special kind of **circular-arc graph** (when no loops are allowed).

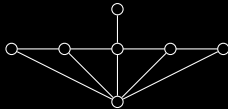
Chudnovsky and Seymour [2005–2012]: “every connected claw-free graph can be obtained from one of the basic claw-free graphs by simple expansion operations,” one of which is a special type of **circular-arc graphs**.

The Problem

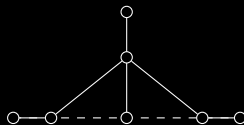
Theorem (Lekkerkerker and Boland 1962). A graph is an interval graph if and only if it doesn't contain any hole ($C_4, C_5, \dots$) or any graph below as an induced subgraph.



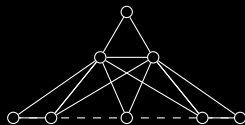
(a) long claw



(b) whipping top

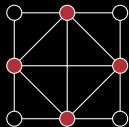


(c) †

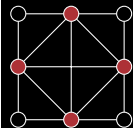
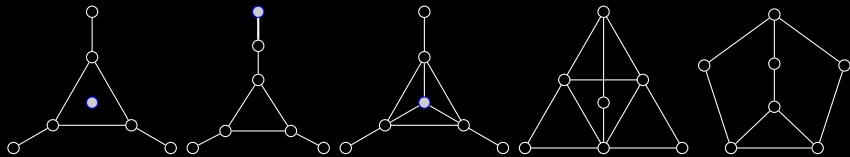


(d) ‡

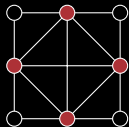
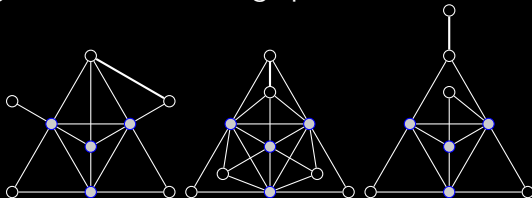
A dashed lines indicate a path of any length.



old $\Rightarrow$ Bold Cold Fold Gold Hold Mold Sold Told olid OLED ould
 日 $\Rightarrow$ 目旦田甲由申旧甲由白电

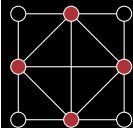


The following are not circular-arc graphs. Which of them are minimal?



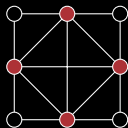
A graph is *chordal* if all induced cycles are triangles.

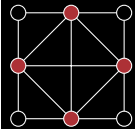
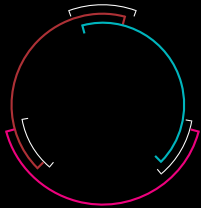
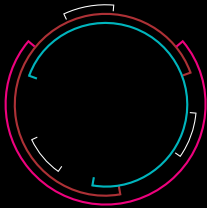
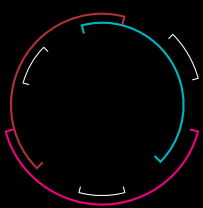
- ▶ chordal: an intersection graph of subtrees of a tree
- ▶ interval = chordal $\cap$ normal Helly circular-arc $\subset$ chordal $\cap$ circular-arc

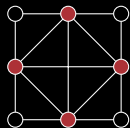
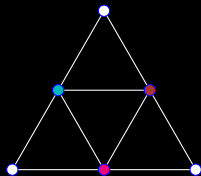
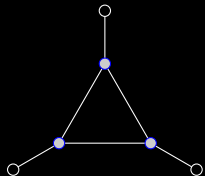
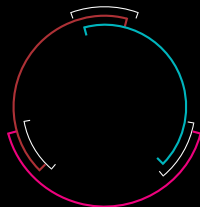
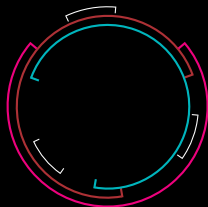
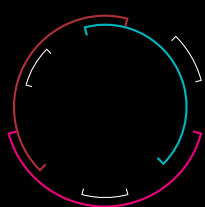


Durán, Grippo, and Safe, *Structural results on circular-arc graphs and circle graphs: A survey and the main open problems*

Problem 1. Give a forbidden induced subgraph characterization for **circular-arc graphs within the class of chordal graphs**. This would extend the characterizations in [Bonomo et al. 2009, Francis et al. 2014] of circular-arc graphs within claw-free chordal graphs and $5K_1$ -free chordal graphs, respectively.

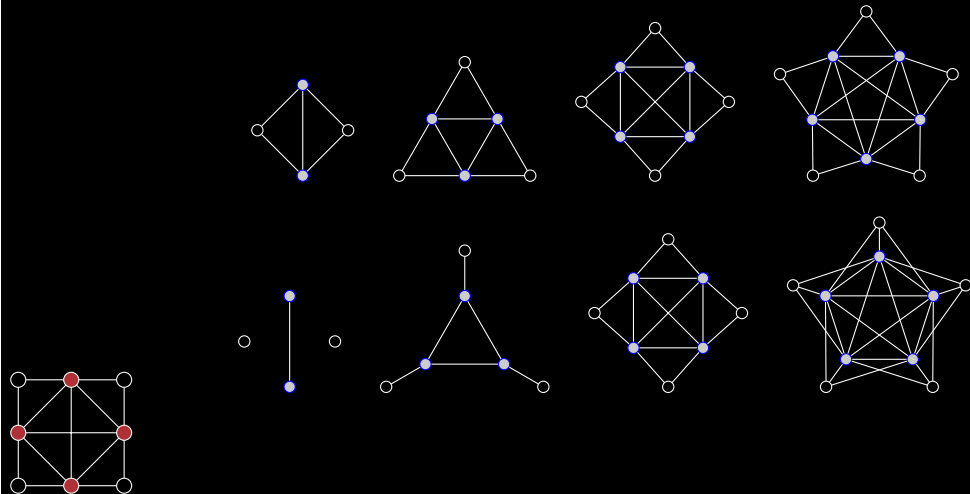




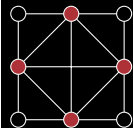


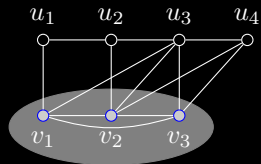
← Does this graph has a unique model (like #1) or multiple ones (like #2)?

k -suns and their complements ($k = 2, 3, 4, 5$)

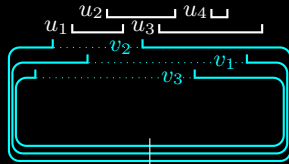


McConnel Flipping

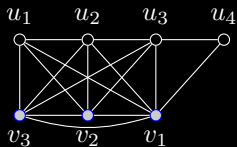




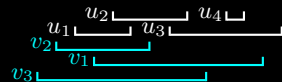
(a)



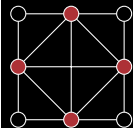
(b)



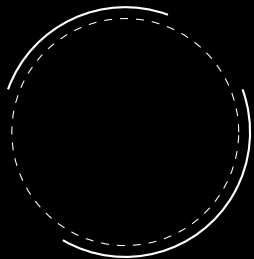
(c)



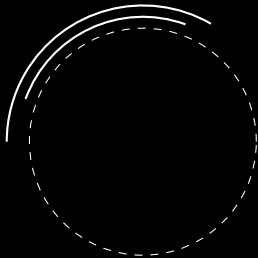
(d)



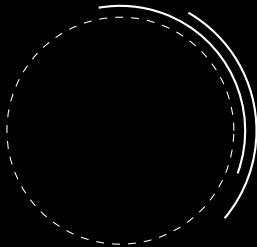
trivial: $a \rightarrow b \rightarrow d \rightarrow c$
 magical: $a \rightarrow c \rightarrow d \rightarrow b$



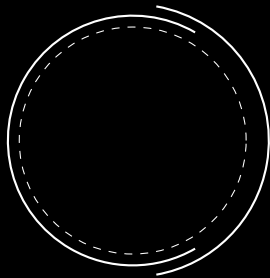
(a) independent



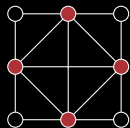
(b) containment



(c) cross



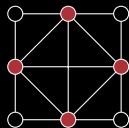
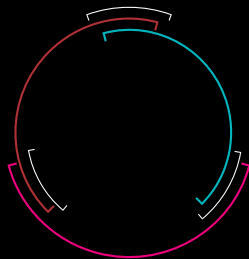
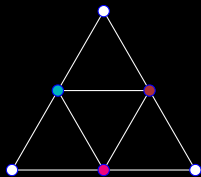
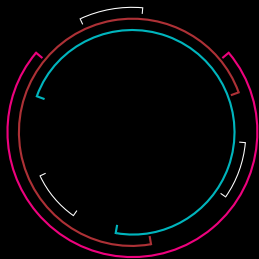
(d) double overlap



Cross-rigid models

A circular-arc model is *cross-rigid* if for every pair v_1 and v_2 .

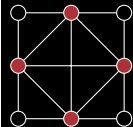
- ▶ The arc $A(v_1)$ contains the arc $A(v_2)$ whenever $N[v_2] \subseteq N[v_1]$.
- ▶ The arcs $A(v_1)$ and $A(v_2)$ double overlap whenever possible.



Which is cross-rigid?
What is a cross-rigid model for P_4 ?

Theorem (Hsu 95).

Any circular-arc graph with no universal vertex admits a cross-rigid model.

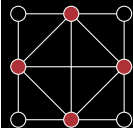


Theorem (McConnel 2001). Circular-arc graphs can be recognized in linear time.

1. find a “suitable” clique K ;
2. “flip” K ;
3. build a special interval model;
4. retrieve a circular-arc model.

$$[lp, rp] \longrightarrow [rp, lp]$$

A crucial trick: we can flip the arcs without the arcs.

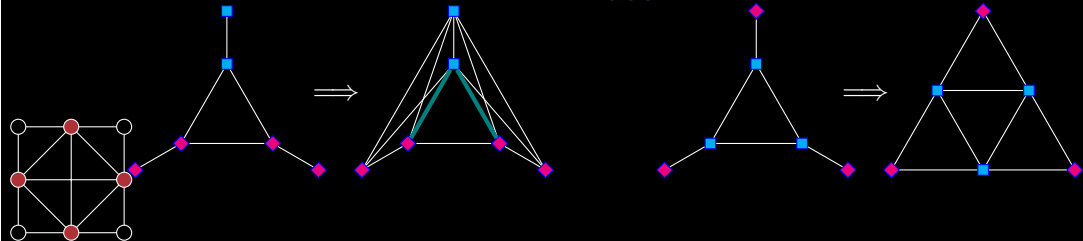


The auxiliary graph G^K

The vertex set is $V(G)$, and a pair of vertices u, v are adjacent in G^K if

- ▶ $u, v \in V(G) \setminus K$ $uv \in E(G)$
- ▶ $u, v \in K$ $N_G(u) \cup N_G(v) \neq V(G)$
- ▶ $u \in K$ and $v \in V(G) \setminus K$ $N_G[v] \not\subseteq N_G[u]$
they are not adjacent, or there exists a vertex adjacent to v but not u in G .

K : blue



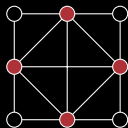
Theorem (C., Krawczyk, 2024⁺).

A chordal graph G is a circular-arc graph if and only if there exists a clique K such that the graph G^K admits an interval model in which for all $x \in K$ and $y \notin K$,

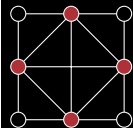
$$I(y) \subseteq I(x) \iff xy \notin E(G).$$

Proof. $\begin{cases} [\text{rp}(x), \text{lp}(x)] & \text{if } x \in K, \\ [\text{lp}(x), \text{rp}(x)] & \text{if } x \notin K. \end{cases}$

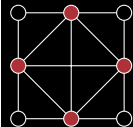
□



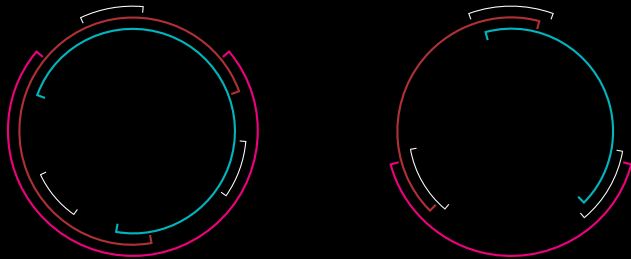
Split $\cap$ Helly Circular-arc



Spinrad: “split graphs ...
often seem to be at the core of algorithms and proofs of difficulty for chordal graphs.”

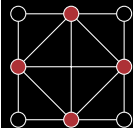


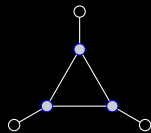
A circular-arc model is *Helly* if the arcs for every maximal clique have a shared point.



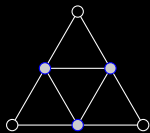
Theorem (Joeris et al. 2011).

All cross-rigid models of Helly circular-arc graphs are Helly.

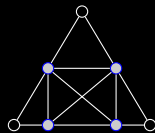




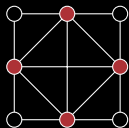
(a) net



(b) sun



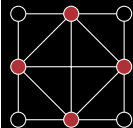
(c) rising sun

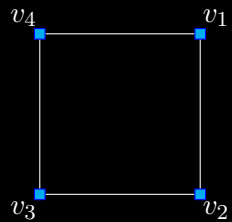
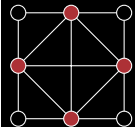


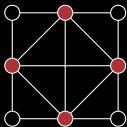
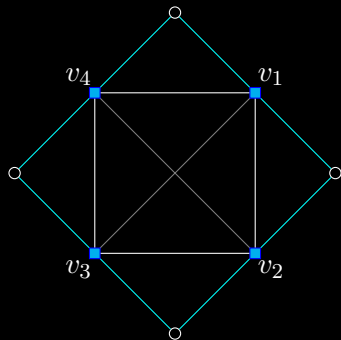
Theorem (C., Krawczyk, 2024⁺).

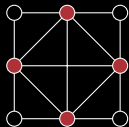
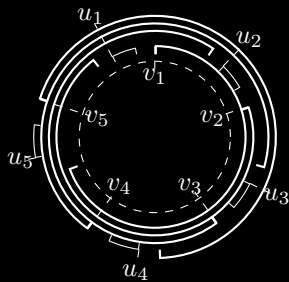
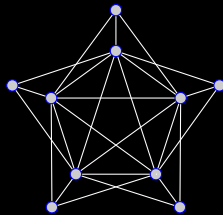
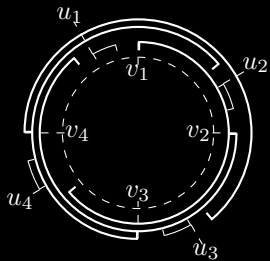
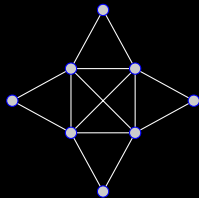
Let G be a split graph with a split partition $K \uplus S$. G is a Helly circular-arc graph if and only if G^K is an interval graph.

Proof. Two vertices $v \in K$ and $u \in V(G) \setminus K$ are adjacent if and only if they are not adjacent in G . □

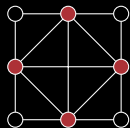
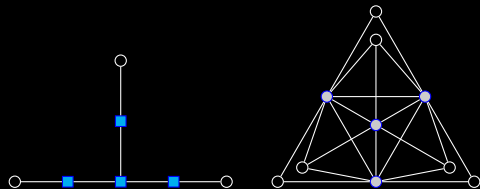
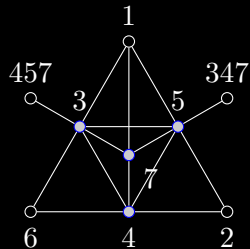
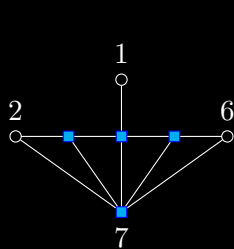




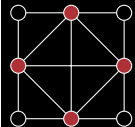


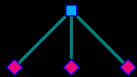


Minimal split graphs that are not Helly circular-arc graphs ($n = 4, 5, \dots$).

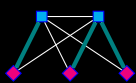


Chordal $\cap$ Circular-arc

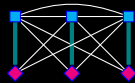




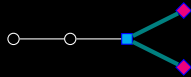
(a)



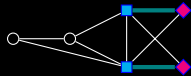
(b)



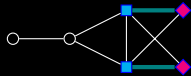
(c)



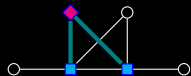
(d)



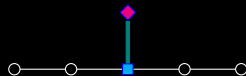
(e)



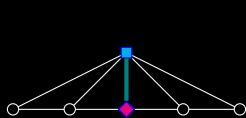
(f)



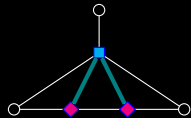
(g)



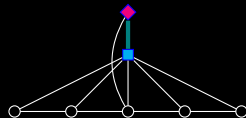
(h)



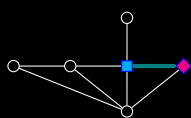
(i)



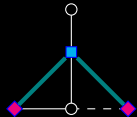
(j)



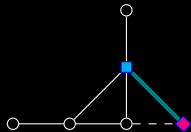
(k)



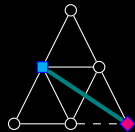
(l)



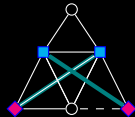
(m)



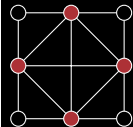
(n)



(o)



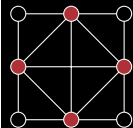
(p)

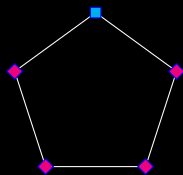


Theorem (C., Krawczyk 2024⁺).

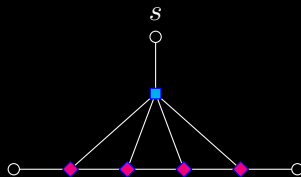
The following are equivalent on a chordal graph G .

- i) The graph G is a circular-arc graph.
- ii) For every simplicial vertex s , the graph $G^{N[s]}$ does not contain any annotated copy of forbidden configurations.
- iii) There exists a simplicial vertex s such that $G^{N[s]}$ does not contain any annotated copy of forbidden configurations.

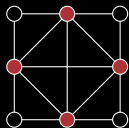




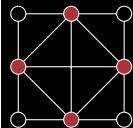
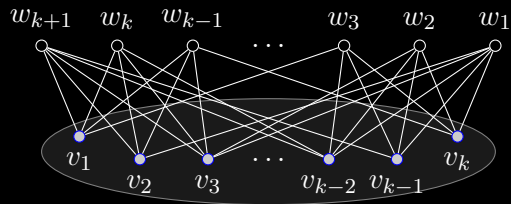
$G^N[s]$



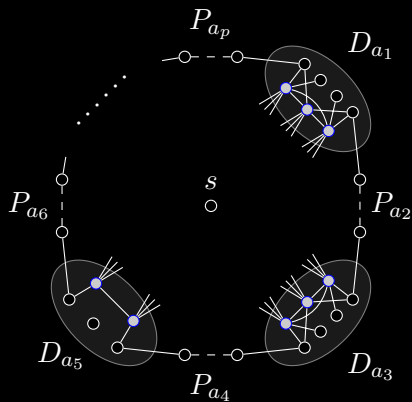
G



The gadget D_k

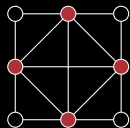


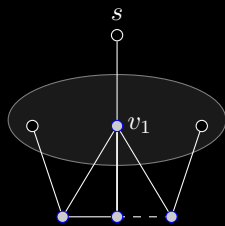
The graph $O(a_1, a_2, \dots, a_p)$



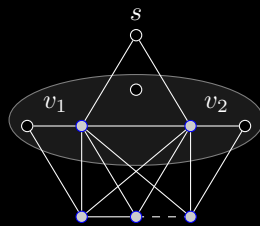
Inside each shadowed ellipse is a gadget.

Each solid vertex is adjacent to all vertices not in the same gadget.

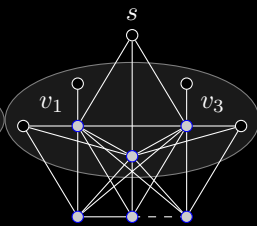




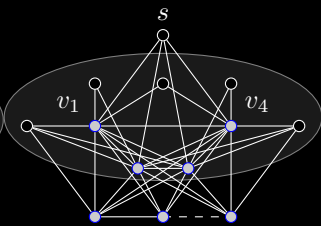
(a) $O(1, q)$



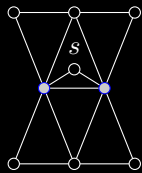
(b) $O(2, q)$



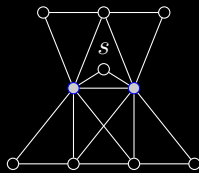
(c) $O(3, q)$



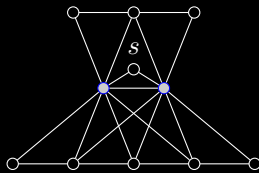
(d) $O(4, q)$



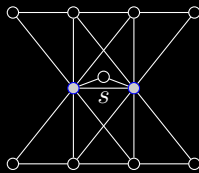
(a) $O(1, 1, 1, 1)$



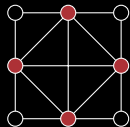
(b) $O(1, 1, 1, 2)$

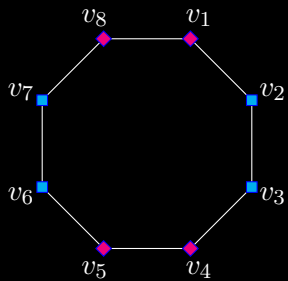
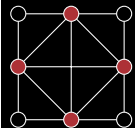


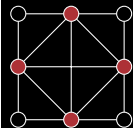
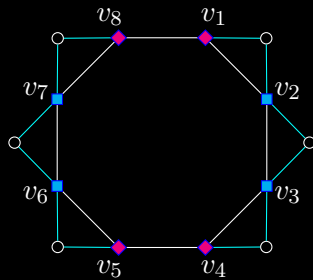
(c) $O(1, 1, 1, 3)$



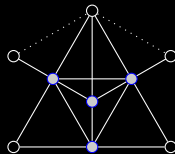
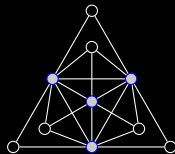
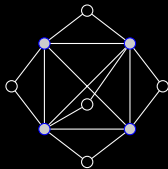
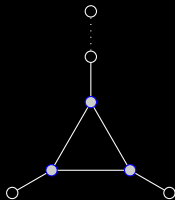
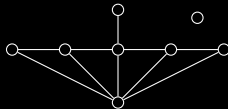
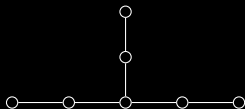
(d) $O(1, 2, 1, 2)$



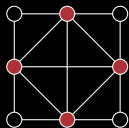




Other minimal chordal forbidden induced subgraphs

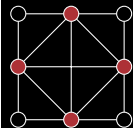


A dotted lines indicate a potential edge (which may or may not be present).



The general case:

- ▶ The forbidden configurations.
- ▶ How to find a “suitable” clique.
- ▶ Labor work.

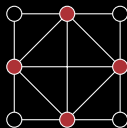


Joint work with Tomasz Krawczyk:

- ▶ Characterization of circular-arc graphs: I. split graphs [arXiv:2403.01947](https://arxiv.org/abs/2403.01947)
- ▶ Characterization of circular-arc graphs: II. McConnell flipping [arXiv:2408.10892](https://arxiv.org/abs/2408.10892)
- ▶ Characterization of circular-arc graphs: III. chordal graphs [arXiv:2409.02733](https://arxiv.org/abs/2409.02733)

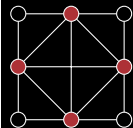
Future work

- ▶ Combine our result with the characterization of Francis, Hell and Stacho [2015].
- ▶ A certifying recognition algorithm.

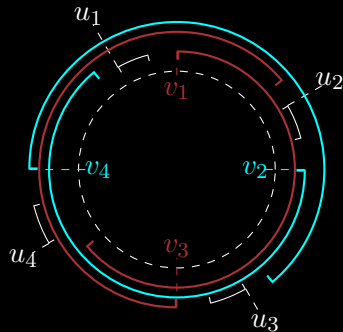


Further open problems

- ▶ The complexity of counting (perfect) matchings on circular-arc graphs.
- ▶ Can circular-arc graphs be canonized in logspace?
 - ▶ Interval graphs [Köbler, Kuhnert, Laubner, Verbitsky, SIAM J. Comput. 2011]
 - ▶ Helly circular-arc graphs [Köbler, Kuhnert, Verbitsky, Inform. Comput. 2016]
 - ▶ C_4 -free circular-arc graphs [Chandoo, Algorithmica 2018]
 - ▶ Bulletin of EATCS, 107:43–71, June 2012



Thanks!



I belong to a clan, or a clique, or a family, or a connection, or whatever you like to call it. *Little Dorrit*

